On a Jaynes-Cummings type model with multiphoton transitions

1986 J. Phys. A: Math. Gen. 19 3607

(http://iopscience.iop.org/0305-4470/19/17/024)

J. Phys. A: Math. Gen. 19 (1986) 3607-3617. Printed in Great Britain

On a Jaynes–Cummings type model with multiphoton transitions

A S Shumovsky, E I Aliskenderov, Fam Le Kien and Nguyen Dinh Vinh

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office, PO Box 79, Moscow, USSR

Received 7 February 1986

Abstract. We present a quantum electrodynamic model, soluble in the dipole and rotating wave approximation, for a three-level atom interacting with a two-mode resonant radiation field through the multiphoton transition mechanism. Population dynamics and photon statistics in this Jaynes-Cummings type model are examined.

1. Introduction

The Jaynes-Cummings model (Jaynes and Cummings 1963) of a two-level atom interacting with a quantised single-mode radiation field is at the core of many problems in quantum optics, NMR and quantum electronics. The importance of this model lies in that it is perhaps the simplest solvable model that describes the essential physics of radiation-matter interaction. Recent studies of this model by Eberly *et al* (1981) and Knight and Radmore (1982) have revealed quantum collapse and revival which are clearly a manifestation of the role of quantum mechanics in the coherence and fluctuation properties of radiation-matter systems. In a series of papers Buck and Sukumar (1981a, b, 1984a, b) and Singh (1982) have proposed three exactly solvable generations of the Jaynes-Cummings model, one involving intensity dependent coupling, one involving multiphoton interaction between field and atom and the other involving the few-level structure of the atom. A generalised model describing a two-mode process in a three-level atom with one-photon transitions has been investigated by Li and Bei (1984) and Bogolubov *et al* (1984, 1985a, b, c, 1986). An excellent review of the dynamical theory of Jaynes-Cummings type models has recently been given by Yoo and Eberly (1985).

The possibility of a multiphoton transition, which proceeds via intermediate states, was first pointed out by Goeppert-Mayer (1931). Various multiphoton transition processes have been studied both theoretically and experimentally. Among them are two-photon and more general multiphoton lasers (McNeil and Walls 1975, Sczaniecky 1980, Gibson and Key 1980, Sharma and Brescansin 1981, Reid *et al* 1981, Zubairy 1982, Wang and Haken 1984a, b), two-photon decay (Tung *et al* 1984, Florescu 1984), multiphoton absorption and emission in a two-level atomic system (Shen 1967, Zubairy and Yeh 1980) and Raman and hyper-Raman processes (Simann 1978, Sainz de los Terreros *et al* 1985).

We wish to present in this paper a rigorous and fully quantum mechanical treatment of multiphoton two-mode processes in a three-level atom on the basis of an exactly solvable Jaynes-Cummings type model.

In § 2 we describe the model. Section 3 contains derivations of general explicit expressions for the time dependence of the level population and photon number operators. In § 4 we study photon statistics. Section 5 gives a consideration of the quantum dressed states and transition probabilities. In § 6 we summarise the results.

2. Description of the model

We consider a three-level atom being at rest in a lossless cavity and interacting with a resonant quantised two-mode radiation field. The energy operator for the atom is

$$H_{\mathrm{A}} = \sum_{j=1}^{3} \hbar \Omega_j R_{jj}. \tag{1}$$

Here the operator $R_{jj} = |j\rangle\langle j|$ describes the population of level j and $\hbar\Omega_j$ is the corresponding level energy. The field Hamiltonian is

$$H_{\mathrm{F}} = \sum_{\alpha=1}^{2} \hbar\omega_\alpha a_\alpha^+ a_\alpha. \tag{2}$$

The photon annihilation and creation operators a_α, a_α^+ $(\alpha = 1, 2)$ describe mode α of the quantised radiation field in the cavity. The ω_α are the mode frequencies. Let the upper level 3 be coupled with the level 1 (level 2) due to the interaction with the field in mode 1 (mode 2) via a m_1 photon (m_2 photon) transition; see figure 1 in which the energy level structure and transition scheme are outlined for the case $m_1 = 3$, $m_2 = 1$. The corresponding multiphoton resonance conditions

$$\Omega_3 - \Omega_\alpha = m_\alpha \omega_\alpha \qquad (\alpha = 1, 2) \tag{3}$$

are assumed to occur. As is well known, the atom–field interaction for a multiphoton process may be described by the effective Hamiltonian where a summation over intermediate states is implicit (Shen 1967, Walls 1971). In the case of the three-level two-mode system considered here the effective Hamiltonian in the electric dipole and rotating wave approximations takes the form

$$H_{\mathrm{AF}} = \hbar \sum_{\alpha=1}^{2} g_\alpha (R_{3\alpha} a_\alpha^{m_\alpha} + R_{\alpha 3} a_\alpha^{+m_\alpha}). \tag{4}$$

Here the operator $R_{ij} = |i\rangle\langle j|$ describes the atomic transition from level j to level i $(i \neq j)$. The mode α atom coupling constant g_α is proportional to $\chi^{(m_\alpha)}$, the dipole

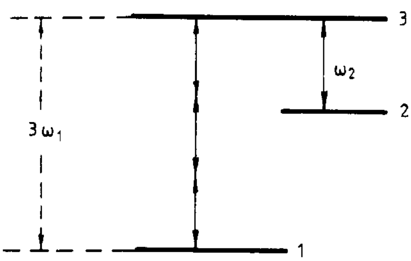

Figure 1. Energy level structure and transition scheme of the system considered in the particular case $m_1 = 3$, $m_2 = 1$.

matrix element for the m_α photon transition between levels 3 and α. The operators $R_{ij} = |i\rangle\langle j|$, $(i, j = 1, 2, 3)$, obey the relations:

$$R_{ij}R_{kl} = R_{il}\delta_{kj} \tag{5a}$$

$$[R_{ij}, R_{kl}] = R_{il}\delta_{kj} - R_{kj}\delta_{il} \tag{5b}$$

$$\sum_{i=1}^{3} R_{ii} = 1. \tag{5c}$$

Thus the full model Hamiltonian of the 'atom-field' system is

$$H = H_A + H_F + H_{AF} = \sum_{j=1}^{3} \hbar\Omega_j R_{jj} + \sum_{\alpha=1}^{2} \hbar\omega_\alpha a_\alpha^+ a_\alpha + \hbar \sum_{\alpha=1}^{2} g_\alpha (R_{3\alpha} a_\alpha^{m_\alpha} + R_{\alpha 3} a_\alpha^{+m_\alpha}). \tag{6}$$

Note that the case $m_1 = m_2 = 1$ has been considered by Bogolubov *et al* (1984, 1985a, b, c, 1986). In the special case when the second mode is excluded from consideration, i.e. when $g_2 = 0$, we can obtain from the Hamiltonian (6) the case examined by Buck and Sukumar (1981b, 1984b) and Singh (1982).

3. Time-dependent level population and photon number operators

3.1. Equations of motion

Starting from the Hamiltonian (6) we write down the Heisenberg equations for various operators in the usual way, i.e. $\dot{\theta} = (\mathrm{i}/\hbar)[H, \theta]$. First of all we define for convenience the subsidiary operators

$$A_\alpha \equiv \mathrm{i}(R_{3\alpha} a_\alpha^{m_\alpha} - R_{\alpha 3} a_\alpha^{+m_\alpha}). \tag{7}$$

Then the Heisenberg equations for the level population operators $R_{\alpha\alpha}$ and the photon number operators $N_\alpha = a_\alpha^+ a_\alpha$ $(\alpha = 1, 2)$ are quickly established:

$$\dot{R}_{\alpha\alpha}(t) = g_\alpha A_\alpha(t) \tag{8a}$$

$$\dot{N}_\alpha(t) = m_\alpha g_\alpha A_\alpha(t). \tag{8b}$$

From these equations it follows that

$$N_\alpha(t) - m_\alpha R_{\alpha\alpha}(t) = \text{constant} \equiv M_\alpha \tag{9}$$

where M_α are constants of motion.

By using relations (5) the Heisenberg equations for A_α are found to be

$$g_\alpha \dot{A}_\alpha(t) = 2g_\alpha^2 \frac{(M_\alpha + m_\alpha)!}{M_\alpha!}[1 - R_{11}(t) - R_{22}(t) - R_{\alpha\alpha}(t)] - g_1 g_2 B(t) \tag{10}$$

where

$$B \equiv R_{21} a_1^{m_1} a_2^{+m_2} + R_{12} a_1^{+m_1} a_2^{m_2}. \tag{11}$$

The operator B obeys the equation of motion

$$\dot{B}(t) = g_1 \frac{(M_1 + m_1)!}{M_1!} A_2(t) + g_2 \frac{(M_2 + m_2)!}{M_2!} A_1(t). \tag{12}$$

Equations (8*a*), (10) and (12) form a closed system of linear equations that has the following integral of motion:

$$g_1 g_2 B(t) - \lambda_1^2 R_{22}(t) - \lambda_2^2 R_{11}(t) = \text{constant} \equiv K. \tag{13}$$

Here the notation

$$\lambda_\alpha^2 = g_\alpha^2 \frac{(M_\alpha + m_\alpha)!}{M_\alpha !} \tag{14}$$

has been introduced.

Let us now differentiate each of equations (8*a*) with respect to time. Taking into account equations (10) and the constant of motion (13) we then obtain

$$\begin{aligned} &\ddot{R}_{11}(t) + (4\lambda_1^2 + \lambda_2^2) R_{11}(t) + 3\lambda_1^2 R_{22}(t) = 2\lambda_1^2 - K \\ &\ddot{R}_{22}(t) + (4\lambda_2^2 + \lambda_1^2) R_{22}(t) + 3\lambda_2^2 R_{11}(t) = 2\lambda_2^2 - K. \end{aligned} \tag{15}$$

Note that equations (15) are the same as the equations obtained previously in the paper of Bogolubov *et al* (1984) for the case $m_1 = m_2 = 1$. One can consider these second-order differential equations as a system of equations for bounded quantum oscillators (Elgin 1980) generating Rabi non-linear oscillations of level populations and photon numbers (Allen and Eberly 1975) in our model. The dependence of (15) upon the numbers of multiple photons per atomic transition m_1 and m_2 is included in the expressions of λ_1, λ_2 and K only.

The solutions of the system (15) can easily be found and represented in the form (Bogolubov *et al* 1984)

$$\begin{aligned} &R_{11}(t) = \mu(\cos\lambda t - 1) + \beta \sin\lambda t + \lambda_1^2[u(\cos 2\lambda t - 1) + v \sin 2\lambda t] + R_{11}(0) \\ &R_{22}(t) = -\mu(\cos\lambda t - 1) - \beta \sin\lambda t + \lambda_2^2[u(\cos 2\lambda t - 1) + v \sin 2\lambda t] + R_{22}(0) \end{aligned} \tag{16}$$

where the operator

$$\lambda \equiv (\lambda_1^2 + \lambda_2^2)^{1/2} = \left(g_1^2 \frac{(M_1 + m_1)!}{M_1 !} + g_2^2 \frac{(M_2 + m_2)!}{M_2 !} \right)^{1/2} \tag{17}$$

describes the Rabi oscillation frequencies. The 'amplitude operators' μ, β, u, v are defined by the initial conditions as follows:

$$\begin{aligned} &\mu = \{\lambda^2[\lambda_2^2 R_{11}(0) - \lambda_1^2 R_{22}(0)] + [\lambda_2^2 - \lambda_1^2]K\}/\lambda^4 \\ &u = \{\lambda^2[1 - 2R_{33}(0)] + K\}/(2\lambda^4) \\ &v = [g_1 A_1(0) + g_2 A_2(0)]/(2\lambda^3) \\ &\beta = [\lambda_2^2 g_1 A_1(0) - \lambda_1^2 g_2 A_2(0)]/\lambda^3. \end{aligned} \tag{18}$$

By using the conservation laws (5*c*) and (9) together with equations (16) we can obtain

$$\begin{aligned} &R_{33}(t) = -\lambda^2[u(\cos 2\lambda t - 1) + v \sin 2\lambda t] + R_{33}(0) \\ &N_1(t) = m_1\{\mu(\cos\lambda t - 1) + \beta \sin\lambda t + \lambda_1^2[u(\cos 2\lambda t - 1) + v \sin 2\lambda t]\} + N_1(0) \\ &N_2(t) = m_2\{-\mu(\cos\lambda t - 1) - \beta \sin\lambda t + \lambda_2^2[u(\cos 2\lambda t - 1) + v \sin 2\lambda t]\} + N_2(0). \end{aligned} \tag{19}$$

Thus we have found the solutions of the equations of motion for the level population and photon number operators in the Heisenberg picture. Since the operators M_2 and the operators λ_α and λ are diagonal in the space of the basis states, we can use solutions (16) and (19) as conventional means to find the time dependence of the level populations and photon numbers. By using these solutions we can also find the statistical characteristics of the photons in the system (see Bogolubov *et al* 1985b and § 4).

3.2. *Time evolution operator*

We denote the free Hamiltonian of the atom and field by H_0:

$$H_0 = H_A + H_F. \tag{20}$$

Then the full Hamiltonian (6) can be written as

$$H = H_0 + H_{AF}. \tag{21}$$

It is easily shown that both H_0 and H_{AF} are constants of motion, i.e.

$$[H, H_0] = [H, H_{AF}] = [H_0, H_{AF}] = 0. \tag{22}$$

This allows the time evolution operator $U(t)$ to be written as

$$U(t) \equiv \exp(-\mathrm{i}Ht/\hbar) = \exp(-\mathrm{i}H_0 t/\hbar)\exp(-\mathrm{i}H_{AF}t/\hbar) = \exp(-\mathrm{i}H_0 t/\hbar)\, U_{\mathrm{int}}(t) \tag{23}$$

where

$$U_{\mathrm{int}}(t) = \exp(-\mathrm{i}H_{AF}t/\hbar) \tag{24}$$

is the time evolution operator in the interaction picture.

By using the identities

$$a_\alpha^{+m} a_\alpha^{m} = \frac{N_\alpha!}{(N_\alpha - m)!} \qquad a_\alpha^{m} a_\alpha^{+m} = \frac{(N_\alpha + m)!}{N_\alpha!} \tag{25}$$

and the relations ($5a$) we can easily show that

$$(H_{AF}/\hbar)^2 = K + \lambda^2 \qquad H_{AF}K = 0 \tag{26}$$

where the constant operators K and λ have been defined in the previous subsection by equations (13) and (17), respectively. From equations (26) it follows that for an integer number $n \geqslant 1$

$$(H_{AF}/\hbar)^{2n} = \frac{K + \lambda^2}{\lambda^2}\lambda^{2n} \qquad (H_{AF}/\hbar)^{2n+1} = (H_{AF}/\hbar)\lambda^{2n}. \tag{27}$$

Hence, it is easy to express the time evolution operators $U_{\mathrm{int}}(t)$ and $U(t)$ in the form

$$U_{\mathrm{int}}(t) = \frac{K + \lambda^2}{\lambda^2}\cos\lambda t - \mathrm{i}(H_{AF}/\hbar)\frac{1}{\lambda}\sin\lambda t - \frac{K}{\lambda^2} \tag{28a}$$

$$U(t) = \exp(-\mathrm{i}H_0 t/\hbar)\, U_{\mathrm{int}}(t). \tag{28b}$$

The time evolution of any operator is now determined by applying the transformation (28) to its value at the initial time $t = 0$. In particular, the density operator $\rho(t)$ of the system 'atom-field' in the Schrödinger picture will be given by

$$\rho(t) = U(t)\rho(0)U^+(t) \tag{29}$$

in terms of its value at time $t = 0$. The density matrix $\rho_F(t)$ of the radiation field and the probability $P(n_1, n_2; t)$ of finding n_1 photons in mode 1 and n_2 photons in mode 2 are found from equation (29) to be

$$\rho_F(t) = \operatorname*{Tr}_A[U(t)\rho(0)U^+(t)] \qquad P(n_1, n_2; t) = \langle n_2, n_1|\rho_F(t)|n_1, n_2\rangle. \tag{30}$$

Using equations (28)-(30) we can examine photon statistics for a given initial state of the system in the manner of Singh (1982).

On the other hand, the time evolution of the operator θ in the Heisenberg picture is given by

$$\theta(t)=U^{+}(t)\theta U(t). \tag{31}$$

Using equations (28) and (31) we can quickly come to the same equations (16)-(19) and examine the time behaviour of the level populations and photon numbers for any initial state of the system.

4. Photon statistics

Let us introduce the following operators of the characteristic function of photon distribution:

$$\chi(\xi_1,\xi_2)=\exp[\mathrm{i}\xi_1 N_1(t)+\mathrm{i}\xi_2 N_2(t)]. \tag{32}$$

Using the conservation laws (9) we find

$$\chi(\xi_1,\xi_2)=\exp(\mathrm{i}\xi_1 M_1+\mathrm{i}\xi_2 M_2)\{[\exp(\mathrm{i}\xi_1 m_1)-1]R_{11}(t)+[\exp(\mathrm{i}\xi_2 m_2)-1]R_{22}(t)+1\}. \tag{33}$$

Denote by $\rho(0)$ the density operator describing an initial state of the 'atom-field' system. Then the characteristic function $\langle\chi(\xi_1,\xi_2)\rangle$ is defined as

$$\langle\chi(\xi_1,\xi_2)\rangle=\operatorname{Tr}\chi(\xi_1,\xi_2)\rho(0). \tag{34}$$

It is connected to the photon distribution function $P(n_1,n_2;t)$ by the relation

$$\langle\chi(\xi_1,\xi_2)\rangle=\sum_{n_1 n_2}\exp(\mathrm{i}\xi_1 n_1+\mathrm{i}\xi_2 n_2)P(n_1,n_2;t) \tag{35}$$

which allows us to get the latter if the former is known.

Once the characteristic and photon distribution functions are known, it is easy to find the statistical moments of photon number $\langle N_\alpha^m(t)\rangle$ and the correlations of modes $\langle N_1^k(t)N_2^l(t)\rangle$ using the relations

$$\begin{aligned}\langle N_\alpha^m(t)\rangle&=\sum_{n_1 n_2}P(n_1,n_2;t)n_\alpha^m=\frac{\partial^m}{\partial(\mathrm{i}\xi_\alpha)^m}\langle\chi(\xi_1=0,\xi_2=0)\rangle\\ \langle N_1^k(t)N_2^l(t)\rangle&=\sum_{n_1 n_2}P(n_1,n_2;t)n_1^k n_2^l=\frac{\partial^{k+l}}{\partial(\mathrm{i}\xi_1)^k\,\partial(\mathrm{i}\xi_2)^l}\langle\chi(\xi_1=0,\xi_2=0)\rangle.\end{aligned} \tag{36}$$

Equations (33)-(36) together with equations (16) allow us to discuss photon statistics for a given initial state of the system. A detailed consideration of this problem will be given below.

We first assume that the atom is initially on a definite level i, i.e.

$$\rho(0)=|i\rangle\langle i|\otimes\rho_{\mathrm{F}} \tag{37}$$

where the density matrix ρ_{F} describes the initial state of the field. Then, by using equations (33), (16) and (37) the characteristic function (34) is found to be

$$\langle\chi(\xi_1,\xi_2)\rangle=\sum_{n_1 n_2}P(n_1,n_2)\exp[\mathrm{i}\xi_1(n_1-m_1\delta_{1i})+\mathrm{i}\xi_2(n_2-m_2\delta_{2i})]\{[\exp(\mathrm{i}\xi_1 m_1)-1]\times R_1(i,n_1,n_2;t)+[\exp(\mathrm{i}\xi_2 m_2)-1]R_2(i,n_1,n_2;t)+1\}. \tag{38}$$

Here $P(n_1, n_2)$ is the initial distribution of photon numbers

$$P(n_1, n_2) = \langle n_2, n_1|\rho_F|n_1, n_2\rangle. \tag{39}$$

The functions $R_\alpha(i, n_1, n_2; t)$ in equation (38) are determined as

$$\begin{aligned} R_1(i, n_1, n_2; t) &= -2\mu(i, n_1, n_2)\sin^2[\lambda(i, n_1, n_2)t/2] \\ &\quad -2\lambda_1^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{1i} \\ R_2(i, n_1, n_2; t) &= 2\mu(i, n_1, n_2)\sin^2[\lambda(i, n_1, n_2)t/2] \\ &\quad -2\lambda_2^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{2i} \end{aligned} \tag{40}$$

where

$$\begin{aligned} \lambda_\alpha(i, n_1, n_2) &= g_\alpha\left(\frac{(n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha)!}{(n_\alpha - m_\alpha\delta_{\alpha i})!}\right)^{1/2} \\ \lambda(i, n_1, n_2) &= [\lambda_1^2(i, n_1, n_2) + \lambda_2^2(i, n_1, n_2)]^{1/2} \\ \mu(i, n_1, n_2) &= 2\lambda_1^2(i, n_1, n_2)\lambda_2^2(i, n_1, n_2)\{\delta_{1i} - \delta_{2i}\}/\lambda^4(i, n_1, n_2) \\ u(i, n_1, n_2) &= [\lambda_1^2(i, n_1, n_2)\delta_{1i} + \lambda_2^2(i, n_1, n_2)\delta_{2i} \\ &\quad - \lambda^2(i, n_1, n_2)\delta_{3i}]/[2\lambda^4(i, n_1, n_2)]. \end{aligned} \tag{41}$$

Comparing equation (38) with equation (35) we obtain

$$\begin{aligned} P(n_1, n_2, t) &= P(n_1 + m_1\delta_{1i} - m_1, n_2 + m_2\delta_{2i})R_1(i, n_1 + m_1\delta_{1i} - m_1, n_2 + m_2\delta_{2i}; t) \\ &\quad + P(n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i} - m_2)R_2(i, n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i} - m_2; t) \\ &\quad + P(n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i})R_3(i, n_1 + m_1\delta_{1i}, n_2 + m_2\delta_{2i}; t) \end{aligned} \tag{42}$$

where

$$R_3(i, n_1, n_2; t) = 2\lambda^2(i, n_1, n_2)u(i, n_1, n_2)\sin^2\lambda(i, n_1, n_2)t + \delta_{3i}. \tag{43}$$

The statistical moments of photon number and the correlations of modes are found from equations (36) and (38) to be

$$\begin{aligned} \langle N_\alpha^m(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)\{(n_\alpha - m_\alpha\delta_{\alpha i})^m + [(n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha)^m \\ &\quad - (n_\alpha - m_\alpha\delta_{\alpha i})^m]R_\alpha(i, n_1, n_2; t)\} \\ \langle N_1^k(t)N_2^l(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)\{(n_1 - m_1\delta_{1i})^k(n_2 - m_2\delta_{2i})^l + (n_1 - m_1\delta_{1i})^k \\ &\quad \times [(n_2 - m_2\delta_{2i} + m_2)^l - (n_2 - m_2\delta_{2i})^l]R_2(i, n_1, n_2; t) \\ &\quad + (n_2 - m_2\delta_{2i})^l[(n_1 - m_1\delta_{1i} + m_1)^k \\ &\quad - (n_1 - m_1\delta_{1i})^k]R_1(i, n_1, n_2; t)\}. \end{aligned} \tag{44}$$

In particular, we find

$$\begin{aligned} \langle N_\alpha(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)[n_\alpha - m_\alpha\delta_{\alpha i} + m_\alpha R_\alpha(i, n_1, n_2; t)] \\ \langle N_\alpha^2(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)\{(n_\alpha - m_\alpha\delta_{\alpha i})^2 + [2m_\alpha(n_\alpha - m_\alpha\delta_{\alpha i}) + m_\alpha^2]R_\alpha(i, n_1, n_2; t)\} \\ \langle N_1(t)N_2(t)\rangle &= \sum_{n_1 n_2} P(n_1, n_2)[(n_1 - m_1\delta_{1i})(n_2 - m_2\delta_{2i}) \\ &\quad + (n_1 - m_1\delta_{1i})m_2R_2(i, n_1, n_2; t) \\ &\quad + (n_2 - m_2\delta_{2i})m_1R_1(i, n_1, n_2; t)]. \end{aligned} \tag{45}$$

Note that in the case $i=1$, $m_1=m_2=1$ equations (45) reduce to the results obtained by Bogolubov *et al* (1985b). Equation (42) for the distribution function of photon numbers can easily be found by other ways using either the time evolution operators (28) and equations (30) in the Schrödinger picture or the dressed state formalism (see § 5). With the aid of equations (42), (44) and (45) we can examine the time behaviour of various photon statistical characteristics, mean photon numbers and mean atomic level populations (Bogolubov 1985c, 1986). In particular, interesting effects such as quantum collapse and revival (Eberly *et al* 1981, Knight and Radmore 1982, Bogolubov *et al* 1986), quantum chaos (Graham and Höhnerbach 1984a, b), and photon antibunching (Bogolubov *et al* 1985c) in exactly soluble models can be investigated.

5. Quantum dressed states and transition probabilities

We represent an eigenstate vector of the free Hamiltonian H_0 by $|i;\, n_1, n_2\rangle$, where $|i\rangle$ is an atomic eigenstate vector corresponding to level i and $|n_1, n_2\rangle$ denotes a Fock state with n_1 photons in mode 1 and n_2 photons in mode 2. This vector describes the so-called undressed state of the system (Haroche 1971, Whitley and Stroud 1976, Radmore and Knight 1982). The eigenstates of the full Hamiltonian H are easily found by solving the stationary Schrödinger equation

$$H\psi = E\psi. \tag{46}$$

Their expressions in terms of the undressed states $|i;\, n_1, n_2\rangle$ are given by

$$\begin{aligned}
\psi_{+;n_1,n_2} &= \frac{\lambda_1(n_1)}{\sqrt{2}\lambda(n_1, n_2)}|1;\, n_1+m_1, n_2\rangle + \frac{\lambda_2(n_2)}{\sqrt{2}\lambda(n_1, n_2)}|2;\, n_1, n_2+m_2\rangle + \frac{1}{\sqrt{2}}|3;\, n_1, n_2\rangle \\
\psi_{-;n_1,n_2} &= \frac{\lambda_1(n_1)}{\sqrt{2}\lambda(n_1, n_2)}|1;\, n_1+m_1, n_2\rangle + \frac{\lambda_2(n_2)}{\sqrt{2}\lambda(n_1, n_2)}|2;\, n_1, n_2+m_2\rangle - \frac{1}{\sqrt{2}}|3;\, n_1, n_2\rangle \\
\psi_{0;n_1,n_2} &= \frac{\lambda_2(n_2)}{\lambda(n_1, n_2)}|1;\, n_1+m_1, n_2\rangle - \frac{\lambda_1(n_1)}{\lambda(n_1, n_2)}|2;\, n_1, n_2+m_2\rangle
\end{aligned} \tag{47}$$

and also by $\psi_{1;\tilde{n}_1 n_2} = |1;\, \tilde{n}_1, n_2\rangle$ with $\tilde{n}_1 \leqslant m_1 - 1$ and $\psi_{2;n_1,\tilde{n}_2} = |2;\, n_1, \tilde{n}_2\rangle$ with $\tilde{n}_2 \leqslant m_2 - 1$. Here for convenience we have put

$$\begin{aligned}
\lambda_1(n_1) &\equiv g_1\left(\frac{(n_1+m_1)!}{n_1!}\right)^{1/2} \\
\lambda_2(n_2) &\equiv g_2\left(\frac{(n_2+m_2)!}{n_2!}\right)^{1/2} \\
\lambda(n_1, n_2) &\equiv [\lambda_1^2(n_1) + \lambda_2^2(n_2)]^{1/2} = \left(g_1^2\frac{(n_1+m_1)!}{n_1!} + g_2^2\frac{(n_2+m_2)!}{n_2!}\right)^{1/2}.
\end{aligned} \tag{48}$$

The eigenenergies $E_{\nu;n_1,n_2}$ ($\nu = 0, \pm, 1, 2$) of the full Hamiltonian H that correspond to the eigenstates $|\psi_{\nu;n_1,n_2}\rangle$ are found to be

$$\begin{aligned}
E_{0;n_1,n_2} &= \hbar(\Omega_3 + n_1\omega_1 + n_2\omega_2) \\
E_{\pm;n_1,n_2} &= E_{0;n_1,n_2} \pm \hbar\lambda(n_1, n_2)
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
E_{1;\tilde{n}_1,n_2} &= \hbar(\Omega_1 + \tilde{n}_1\omega_1 + n_2\omega_2) \qquad \text{where } \tilde{n}_1 \leqslant m_1 - 1 \\
E_{2;n_1,\tilde{n}_2} &= \hbar(\Omega_2 + n_1\omega_1 + \tilde{n}_2\omega_2) \qquad \text{where } \tilde{n}_2 \leqslant m_2 - 1.
\end{aligned} \tag{50}$$

Thus the spectrum of the Hamiltonian H consists of a lattice of triplets of closely spaced eigenstates $\psi_{s;n_1,n_2}(s=0,\pm)$ and two sets of equally spaced undressed states $|1;\tilde{n}_1,n_2\rangle$ with $\tilde{n}_1 \leq m_1-1$ and $|2;n_1,\tilde{n}_2\rangle$ with $\tilde{n}_2 \leq m_2-1$. Each triplet is characterised by a pair of indices (n_1,n_2) that indicate that those triplet states are linear combinations of the three degenerate states $|1;n_1+m_1,n_2\rangle$, $|2;n_1,n_2+m_2\rangle$ and $|3;n_1,n_2\rangle$, see (33). The energy splittings $\pm\hbar\lambda(n_1,n_2)$ within the triplet (n_1,n_2) are of course due to the coupling of the atom to the field and are referred to as the resonant Stark effect. The triplet eigenstates $\psi_{s;n_1,n_2}(s=0,\pm)$ are called quantum dressed states of the system (Haroche 1971, Whitley and Stroud 1976, Radmore and Knight 1982, Yoo and Eberly 1985). It is interesting to note that the dressed states $\psi_{0;n_1,n_2}$ (see the last equation in (47)) are the coherent superpositions of only the undressed states $|1;n_1+m_1,n_2\rangle$ and $|2;n_1,n_2+m_2\rangle$ but not $|3;n_1,n_2\rangle$. The existence of such dressed states uncoupled with the upper level 3 plays the important role in the mechanism of the population trapping effect (Radmore and Knight 1982, 1984, Dalton and Knight 1982, Yoo and Eberly 1985) due to which the decay channels in multiphoton excitation can be turned off.

We now proceed to calculate the probabilities for the multiphoton transitions of the atom. Let us denote by $\varphi(t)$ the wavefunction of the total system 'atom+field' in the Schrödinger picture. Then the probability of finding the atom on its jth level at time t as a result of the transition $i\to j$ initiated by n_1 photons in mode 1 and n_2 photons in mode 2 of the field is defined by the formula

$$P(t;i\to j)=\sum_{n_1',n_2'}|\langle\varphi_{in_1n_2}(t)|j;n_1',n_2'\rangle|^2. \tag{51}$$

Here the initial condition

$$\varphi_{in_1n_2}(0)=|i;n_1,n_2\rangle \tag{52}$$

has been assumed. By expanding $\varphi_{in_1n_2}(0)$ in terms of the dressed eigenstates (47) we can easily find the time-dependent wavefunctions $\varphi_{in_1n_2}(t)$. They are

$$\begin{aligned}
\varphi_{1n_1+m_1n_2}(t)\exp[\mathrm{i}(\Omega_3+n_1\omega_1+n_2\omega_2)t]&=|1;n_1+m_1,n_2\rangle\\
&\quad\times\{\lambda_1^2(n_1)\cos[\lambda(n_1,n_2)t]+\lambda_2^2(n_2)\}/\lambda^2(n_1,n_2)\\
&\quad+|2;n_1,n_2+m_2\rangle\{\cos[\lambda(n_1,n_2)t]-1\}\lambda_1(n_1)\lambda_2(n_2)/\lambda^2(n_1,n_2)\\
&\quad-i|3;n_1,n_2\rangle\lambda_1(n_1)\sin[\lambda(n_1,n_2)t]/\lambda(n_1,n_2)\\
\varphi_{2n_1n_2+m_2}(t)\exp[\mathrm{i}(\Omega_3+n_1\omega_1+n_2\omega_2)t]&=|1;n_1+m_1,n_2\rangle\\
&\quad\times\{\cos[\lambda(n_1,n_2)t]-1\}\lambda_1(n_1)\lambda_2(n_2)/\lambda^2(n_1,n_2)\\
&\quad+|2;n_1,n_2+m_2\rangle\{\lambda_2^2(n_2)\cos[\lambda(n_1,n_2)t]+\lambda_1^2(n_1)\}/\lambda^2(n_1,n_2)\\
&\quad-i|3;n_1,n_2\rangle\lambda_2(n_2)\sin[\lambda(n_1,n_2)t]/\lambda(n_1,n_2)\\
\varphi_{3n_1n_2}(t)\exp[\mathrm{i}(\Omega_3+n_1\omega_1+n_2\omega_2)t]&=-i|1;n_1+m_1,n_2\rangle\\
&\quad\times\lambda_1(n_1)\sin[\lambda(n_1,n_2)t]/\lambda(n_1,n_2)\\
&\quad-i|2;n_1,n_2+m_2\rangle\lambda_2(n_2)\sin[\lambda(n_1,n_2)t]/\lambda(n_1,n_2)\\
&\quad+|3;n_1,n_2\rangle\cos[\lambda(n_1,n_2)t].
\end{aligned} \tag{53}$$

Hence the expressions of the probabilities (51) for the multiphoton transitions are found to be

$$P(t;i\to j)=R_j(i,n_1,n_2;t) \tag{54}$$

where the functions $R_j(i, n_1, n_2; t)$ have been defined by equations (40) and (43). Equation (54) implies that the transition probability $P(t; i \to j)$ is equal to the population of level j under the initial state (52). Using equation (54) and the detailed balance principle, under the initial condition (37) we can easily obtain the same equation (42) for the photon distribution function $P(n_1, n_2; t)$.

6. Summary

In this paper we have presented and studied a soluble Jaynes–Cummings type model. The model considered consists of a lambda configuration three-level atom interacting with a two-mode resonant radiation field through the multiphoton transition mechanism. The general explicit expressions for the time-dependent level population and photon number operators have been derived in various ways using either equations of motion or time evolution operators. The quantum electrodynamic expression of Rabi oscillation frequencies has been obtained. Photon statistics in the model has been studied. Expressions for the photon distribution, characteristic function, mean photon numbers, statistical moments and correlations of photon numbers in the modes are presented for various initial conditions. The quantum dressed eigenstates and the energy spectrum have been found. The probabilities for multiphoton transitions from one level to another level of the atom have been calculated. Application of the model to the study of multiphoton two-model lasers will be discussed in a future work.

References

Allen L and Eberly J H 1975 *Optical resonance and two-level atoms* (New York: Wiley)
Bogolubov N N Jr, Fam Le Kien and Shumovsky A S 1984 *Phys. Lett.* **101A** 201
—— 1985a *Phys. Lett.* **107A** 173
—— 1985b *Phys. Lett.* **107A** 456
—— 1985c *Preprint, JINR* E17-85-402
—— 1986 *J. Phys. A: Math. Gen.* **19** 191
Buck B and Sukumar C V 1981a *Phys. Lett.* **81A** 132
—— 1981b *Phys. Lett.* **83A** 211
—— 1984a *J. Phys. A: Math. Gen.* **17** 877
—— 1984b *J. Phys. A: Math. Gen.* **17** 885
Dalton B J and Knight P L 1982 *J. Phys. B: At. Mol. Phys.* **15** 3997
Eberly J H, Narozhny N B and Sanchez-Mondragon J J 1981 *Phys. Rev.* A **23** 236
Elgin J N 1980 *Phys. Lett.* **80A** 140
Florescu V 1984 *Phys. Rev.* A **30** 2441
Gibson A F and Key M H 1980 *Rep. Prog. Phys.* **43** 3
Goeppert-Mayer M 1931 *Ann. Phys., Lpz* **9** 273
Graham R and Höhnerbach M 1984a *Z. Phys.* **B 57** 233
—— 1984b *Acta Phys. Aust.* **56** 45
Haroche S 1971 *Ann. Phys., Paris* **6** 189, 327
Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
Knight P L and Radmore P M 1982 *Phys. Lett.* **90A** 342
Li X and Bei N 1984 *Phys. Lett.* **101A** 169
McNeil K J and Walls D F 1975 *J. Phys. A: Math. Gen.* **8** 104
Radmore P M and Knight P L 1982 *J. Phys. B: At. Mol. Phys.* **15** 561
—— 1984 *Phys. Lett.* **102A** 180
Reid M, McNeil K J and Walls D F 1981 *Phys. Rev.* A **24** 2029
Sainz de los Terreros L, Santos M and Gonzalez-Diaz P F 1985 *Phys. Rev.* A **31** 1598

Sharma M P and Brescansin L M 1981 *Phys. Rev.* A **23** 1893
Shen Y R 1967 *Phys. Rev.* **155** 921
Sczaniecky L 1980 *Opt. Acta* **27** 251
Simann H D 1978 *J. Phys. A: Math. Gen.* **11** 1799
Singh S 1982 *Phys. Rev.* A **25** 3206
Tung J H, Ye X M, Salamo G J and Chan F T 1984 *Phys. Rev.* A **30** 1175
Walls D F 1971 *J. Phys. A: Math. Gen.* **4** 813
Wang Z C and Haken H 1984a *Z. Phys.* B **55** 361
—— 1984b *Z. Phys.* B **56** 77
Whitley R M and Stroud C R Jr 1976 *Phys. Rev.* A **14** 1498
Yoo H I and Eberly J H 1985 *Phys. Rep.* **118** 239
Zubairy M S 1982 *Phys. Lett.* **87A** 162
Zubairy M S and Yeh J J 1980 *Phys. Rev.* A **21** 1624